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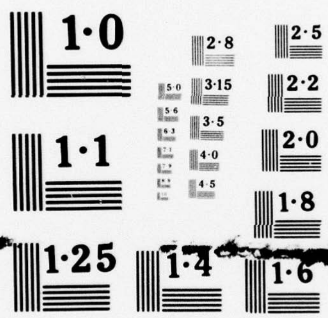

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FURTHER RESULTS ON THE UNCERTAINTY THRESHOLD PRINCIPLE*

by

Richard T. Ku**
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Abstract

Additional quantitative results are presented for the existence of optimal decision rules and stochastic stability for linear systems with white random parameters with respect to quadratic performance criteria, by examining a specific version of a multivariable optimization problem.

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1. INTRODUCTION

This paper considers the optimal stochastic control of a multivariable linear system, with a specific structure, with respect to a quadratic index of performance. The system dynamics are described by a linear vector difference equation in which both the system matrix ($\underline{A}$) and control matrix ($\underline{B}$) are multiplied by white, possible correlated, scalar random sequences.

A threshold condition involving the maximum eigenvalue of the system matrix $\underline{A}$ and the means, variances, and cross correlations of the white parameters is obtained. If the threshold condition is violated, then there does not exist an optimal solution to the infinite horizon optimization problem, and the resultant closed-loop system is not stable in a mean square sense.

The results of this paper represent another manifestation of the Uncertainty Threshold Principle (UTP) reported by Athans, Ku, and Gershwin in [1] and represents a specific multivariable extension of the scalar results reported in [1]. It also generalizes the results of Kutayama [2], which dealt with control-dependent white noise, to the case of simultaneously, possibly correlated, state- and control-dependent white noise parameters.

2. PROBLEM FORMULATION

Consider a linear stochastic discrete-time system whose dynamics are described by the following vector difference equation

$$\underline{x}(t+1) = \gamma(t)\underline{A}\ \underline{x}(t) + \delta(t)\underline{B}\ \underline{u}(t) + \underline{\xi}(t) \tag{1}$$

where $\underline{x}(t)$ is the n-dimensional state vector, $\underline{u}(t)$ is the m-dimensional control vector, and $\underline{\xi}(t)$ is white gaussian process noise. Assume that $\underline{A}$ and $\underline{B}$ are constant matrices of appropriate dimensions and that $[\underline{A}, \underline{B}]$ is a controllable pair.

Further assume that the <u>scalars</u> $\gamma(t)$ and $\delta(t)$ are gaussian white random sequences (uncorrelated in time) with known stationary statistics. More precisely, we assume

$$E\{\gamma(t)\} = \bar{\gamma}\ ;\ E\{(\gamma(t) - \bar{\gamma})(\gamma(\tau) - \bar{\gamma})\} = \Gamma\ \delta(t,\tau) \tag{2}$$

$$E\{\delta(t)\} = \bar{\delta}\ ;\ E\{(\delta(t)-\bar{\delta})(\delta(\tau) - \bar{\delta})\} = \Delta\ \delta(t,\tau) \tag{3}$$

$$E\{(\gamma(t) - \bar{\gamma})(\delta(\tau) - \bar{\delta})\} = \Lambda\ \delta(t,\tau) \tag{4}$$

$$E\{\underline{\xi}(t)\} = \underline{0}\ ;\ E\{\underline{\xi}(t)\underline{\xi}'(\tau)\} = \underline{\Xi}\ \delta(t,\tau) \tag{5}$$

where $\delta(t,\tau)$ is the Kroenecker delta ($\delta(t,\tau) = 1$ if $t = \tau$, $\delta(t,\tau) = 0$ if $t \neq \tau$). Furthermofre assume that the process noise $\underline{\xi}(t)$ is mutually independent of the random parameters $\gamma(t)$ and $\delta(t)$.

We consider a standard quadratic cost functional

$$J = E\{\frac{1}{T} \sum_{t=0}^{T} \underline{x}'(t)\underline{Q}\ \underline{x}(t) + \underline{u}'(t)\underline{R}\ \underline{u}(t)\} \tag{6}$$

where $\underline{Q}$ is positive semidefinite, $\underline{R}$ is positive definite, and $[\underline{Q}^{1/2}, \underline{A}]$ is an observable pair.

Under the assumptions that we can measure the entire state vector $\underline{x}(t)$ exactly, at each instate of time, we wish to find the causal optimal control sequence $\underline{u}(0)$, $\underline{u}(1)$, $\underline{u}(t)$, ... which minimizes the quadratic cost (6).

Remark 1 The problem considered by Katayama [2] corresponds to the case

$$\bar{\gamma} = 1, \ \Gamma = 0, \ \Lambda = 0, \ \underline{\Xi} = \underline{0} \tag{7}$$

3. SOLUTION

The problem can be readily solved by stochastic dynamic programming [3] for any fixed value of the planning horizon time T. The derivation is straightforward and hence omitted. Only the results are stated.

The optimal control is obtained using linear state variable feedback, i.e.

$$\underline{u}^*(t) = -\underline{G}(t)\underline{x}(t) \tag{8}$$

The optimal $m \times n$ feedback control gain matrix $\underline{G}(t)$ is given by the formula

$$\underline{G}(t) = [\underline{R} + (\bar{\delta}^2 + \Delta)\underline{B}'\underline{K}(t+1)\underline{B}]^{-1}(\bar{\gamma}\,\bar{\delta} + \Lambda)\underline{B}'\underline{K}(t+1)\underline{A} \tag{9}$$

The $n \times n$ matrix $\underline{K}(t)$ satisfies a recursive matrix equation of the form

$$\begin{aligned}\underline{K}(t) = {} & (\bar{\gamma}^2 + \Gamma)\underline{A}'\underline{K}(t+1)\underline{A} + \underline{Q} \\ & - (\bar{\gamma}\,\bar{\delta} + \Lambda)^2\underline{A}'\underline{K}(t+1)\underline{B}[\underline{R} + (\bar{\delta}^2 + \Delta)\underline{B}'\underline{K}(t+1)\underline{B}]^{-1}B'\underline{K}(t+1)\underline{A}\end{aligned} \tag{10}$$

with $\underline{K}(T) = \underline{0}$.

Remark 2 The recursion (10) will be referred to as the UTP matrix equation; it is similar to a matrix Riccati equation. However, unlike Riccati equations it cannot be related to a coupled set of linear equations. Under our assumption the positive definite matrix $\underline{K}(t)$ exists and is bounded for all finite planning horizon times T.

The optimal cost (6) is given by

$$J^*(\underline{x}(0),T) = \frac{1}{T}\,\underline{x}'(0)\underline{K}(0)\underline{x}(0) + \frac{1}{T}\sum_{t=0}^{T}\underline{K}(t)\underline{\Xi} \tag{11}$$

4. THE INFINITE HORIZON CASE ($T \to \infty$)

The interesting results occur as one analyzes the infinite horizon case, $T \to \infty$, so as to examine the existence of an optimal solution and the stabilizability of the stochastic system [1], [2]. Once more we shall show that there exists a threshold condition which provides a dividing line between existence and non-existence of optimal solutions to the problem as $T \to \infty$. We summarize the main result as follows.

<u>Theorem 1</u> (Uncertainty Threshold Principle)

An optimal solution exists for the problem stated in Section 2 as $T \to \infty$ <u>if</u> <u>and</u> <u>only</u> <u>if</u>

$$\max_i |\lambda_i(\underline{A})| < 1/\beta \qquad ; \; i = 1,2,\ldots,n \tag{12}$$

where β is defined by

$$\beta^2 = \bar{\gamma}^2 + \Gamma - \frac{(\bar{\gamma}\,\bar{\delta} + \Lambda)^2}{\bar{\delta}^2 + \Delta} \geq 0 \tag{13}$$

and $\max_i |\lambda_i(\underline{A})|$ denotes the magnitude of the maximum eigenvalue of the constant system matrix $\underline{A}$ in the system dynamics (1).

Before we present the proof of the theorem it is important to make some remarks.

<u>Remark 3</u> In the case of non-random parameters ($\Gamma = \Delta = \Lambda = 0$), $\beta = 0$. This means that given our assumptions of $[\underline{A}, \underline{B}]$ controllability and $[\underline{A}, \underline{Q}^{1/2}]$ observability, one can always solve the infinite horizon optimal control problem independent of the (open loop) eigenvalues of $\underline{A}$. On the other hand, as the variances Γ and Δ of the random parameters increase, then β increases, and the value of $1/\beta$ defines the radius of a shrinking disc which must contain all the open-loop eigenvalues of $\underline{A}$ in order for the problem to have a solution.

<u>Remark 4</u> If the condition (12) is violated, i.e. if

$$\max_i |\lambda_i(\underline{A})| \geq 1/\beta \tag{14}$$

then there is no solution to the optimal control problem, and one cannot stabilize (in a mean square sense) the system (1). Under these conditions the optimal cost, J^*, as defined by (11) undergoes exponential growth as T increases

$$J^*(T) \geq c\, e^{\max_i |\lambda_i(\beta \underline{A})| T} \quad ; \; c = \text{constant} \tag{15}$$

Because of the explosive growth of the optimal cost, then only short term (small T) decisions make sense; see also [1].

As in the scalar case [1], even if condition (14) holds, the control gain matrix $\underline{G}(t)$, see eq. (9), remains well behaved and bounded

$$\underline{G} = \lim_{||\underline{K}(t+1)|| \to \infty} \frac{(\overline{\gamma\delta} + \Lambda)}{\bar{\delta}^2 + \Delta} [\underline{B}'\underline{K}(t+1)\underline{B}]^{-1}\underline{B}'\underline{K}(t+1)\underline{A} \tag{16}$$

Next we present the details of proving Theorem 1. We remark that the proof essentially uses algebraic manipulations and known properties of discrete Lyapunov and Riccati matrix equations.

The main idea of the proof is to examine the behavior of $\lim_{T\to\infty} K(t)$, or the behavior "backward in time" of the UTP matrix equation. The arguments are similar, but not identical, to those given in [2].

For the sake of notational convenience define the scalars

$$\alpha_1 \triangleq \bar{\gamma}^2 + \Gamma; \alpha_2 = (\bar{\gamma}\,\bar{\delta} + \Lambda)^2;\ \alpha_3 \triangleq 1/\bar{\delta}^2 + \Delta \tag{17}$$

Then the UTP equation (10) can be written as

$$\begin{aligned}\underline{K}(t) &= \alpha_1\underline{A}'\underline{K}(t+1)\underline{A} + \underline{Q} \\ &\quad - \alpha_2\underline{A}'\underline{K}(t+1)\underline{B}[\underline{R} + \frac{1}{\alpha_3}\underline{B}'\underline{K}(t+1)\underline{B}]^{-1}\underline{B}'\underline{K}(t+1)\underline{A}\end{aligned} \tag{18}$$

From Eqs. (13) and (17) one sees that

$$\beta^2 = \alpha_1 - \alpha_2\alpha_3 \tag{19}$$

By adding and subtracting

$$\alpha_2\alpha_3\underline{A}'\underline{K}(t+1)\underline{A} \tag{20}$$

to the right hand side of Eq. (18), and some algebra, Eq. (10) reduces to

$$\begin{aligned}\underline{K}(t) &= \beta^2\underline{A}'\underline{K}(t+1)\underline{A} + \underline{Q} \\ &\quad + \alpha_2\alpha_3\underline{A}'[\underline{K}(t+1) - \underline{K}(t+1)\underline{B}[\alpha_3\underline{R} + \underline{B}'\underline{K}(t+1)\underline{B}]^{-1}\underline{B}'\underline{K}(t+1)]\underline{A}\end{aligned} \tag{21}$$

Attention is focused to the matrix

$$\underline{M}(t+1) \triangleq \underline{K}(t+1) - \underline{K}(t+1)\underline{B}[\alpha_3\underline{R} + \underline{B}'\underline{K}(t+1)\underline{B}]^{-1}\underline{B}'\underline{K}(t+1) \tag{22}$$

Such matrices arise naturally in the Riccati equation of standard linear quadratic problems where the control weighting matrix is $\alpha_3\underline{R}$. Under the given assumptions of $[\underline{A}, \underline{B}]$ controllability and $[\underline{A}, \underline{Q}^{1/2}]$ observability it is known that that [4],[5]*

(a) $\underline{M}(t+1) = \underline{M}'(t+1) > \underline{0}$ (23)

(b) There exists a bound

$$\underline{L} \geq \underline{M}(t) \quad \text{all } t \tag{24}$$

Since $\underline{M}(t+1)$ is positive definite, so is $\alpha_2\alpha_3\underline{A}'\underline{M}(t+1)\underline{A}$. Hence we obtain

$$\underline{K}(t) \geq \beta^2\underline{A}'\underline{K}(t+1)\underline{A} + \underline{Q} \tag{25}$$

From (25) it is obvious that if any eigenvalue of $(\beta\ \underline{A})$ is greater than unity then $\underline{K}(t)$ grows without bound backward in time, that $\lim_{T\to\infty} \underline{K}(t)$ does not exist and that the optimal cost undergoes exponential growth as indicated by (15). On the other hand from (24) and (25) one obtains that

$$\underline{K}(t) \leq \beta^2\underline{A}'\underline{K}(t+1)\underline{A} + \underline{Q} + \alpha_2\alpha_3\underline{A}'\underline{L}\ \underline{A} \tag{26}$$

Hence if all eigenvalues of $(\beta\ \underline{A})$ are less than unity, the right hand side of the recursion (26) will approach a bounded constant solution matrix and so will $\underline{K}(t)$. Hence, the limiting solution $\lim_{T\to\infty} \underline{K}(t)$ is well defined.

We remark that the above proof requires that $\underline{B}$ is n×n and nonsingular, as required in the Corollary of [2]. However we believe that this is a sufficient, but by no means necessary, condition; it could probably

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* The notation $\underline{A} > \underline{B}$ ($\underline{A} \geq \underline{B}$) means that $\underline{A}-\underline{B}$ is positive definite ($\underline{A} - \underline{B}$ is positive semidefinite).

be removed by a more detailed analysis of the UTP difference equation.

5. CONCLUSIONS

The quantitative results of the Uncertainty Threshold Principle have been extended to a special case of a multivariable control problem, generalizing the results in [1] and [2].

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Additional quantitative results are presented for the existence of optimal decision rules and stochastic stability for linear systems with white random parameters with respect to quadratic performance criteria, by examining a specific version of a multivariable optimization problem.

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